The moving semibounded magnetoactive plasma in field of a flat gravitational wave

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Abstract

The problem of a moving semibounded magnetoactive plasmas in a plane gravitational wave is research on the basis of the self-consistent equations, obtained earlier by the Authors.

1 Introduction

In the series of articles ([1], [2], [3], [5]), based on exact stationary1 solutions of the set of Maxwell equations and the equations of relativistic magnetohydrodynamics for an unlimited homogeneous plasma, we established an essentially nonlinear behaviour of a highly magnetized plasma in the field of gravitational radiation propagating across the magnetic field. On reaching some critical conditions, the plasma begins accelerating up to subluminal velocities in the direction of the gravitational wave propagation. Simultaneously, the magnetic field strength greatly increases. This article is devoted to an analysis of the boundary-value problem for a semibounded plasma, with a goal to study the mechanism of the origin of a gravimagnetic shock wave.

2 Self-consistent equations of motion of a magneto active plasma in the PGW field

In Ref. [1], from the condition of coincidence of dynamic velocities in the energy-momentum tensors (EMT) of a perfect fluid and the electromagnetic field, the equations of relativistic magnetohydrodynamics of plasma in the arbitrary gravitational field were obtained, and their exact solution was found in the background of the plane gravitational wave (PGW) metric, corresponding to an initial state of a homogeneous electroneutral plasma with a frozen-in homogeneous magnetic field.

A general property of the above solution is the presence of a singularity on the hypersurface

$$\Delta(u) = 1 - \alpha^2 \left[e^{2\beta(u)} - 1\right] = 0, \tag{1}$$

$$\alpha^2 = H_0^2 \sin^2 \Omega / 4\pi(\varepsilon_0 + p_0) \tag{2}$$

is a dimensionless parameter while $\beta(u)$ is an arbitrary function of the retarded time u (the PGW amplitude). This singularity in a weak PGW ($\beta \ll 1$) is

reached under the condition:

$$2\beta(u)\alpha^2 \geqslant 1\,. \tag{3}$$

An analysis of the solution shows that, when the condition (3) holds, a shock wave is formed in the plasma, propagating with a subluminal velocity in the direction of PGW propagation. A necessary condition for the excitation of a shock magnetohydrodynamic wave (GMSW) is a high magnetization of the plasma:

$$\alpha^2 \gg 1\ . \tag{4}$$

However, the above solution of the RMHD equations is essentially stationary (it depends only on u, the retarded time) and corresponds to an initially homogeneous magnetoactive plasma. Therefore the solution obtained in Ref. [3]] cannot describe the dynamics of the shock wave excitation mechanism.

In this paper, we will study the problem of PGW distribution in an isotropic magnetoactive plasma with boundary conditions, supposing that the null hypersurface:

$$\Sigma_0 : u = 0 \tag{5}$$

is the surface of PGW front, i.e., the PGW is absent at $u \leqslant 0$, –

$$\beta(u)_{|u\leqslant 0} = 0\,; \qquad \beta'(u)_{|u\leqslant 0} = 0\,; \qquad L(u)_{|u\leqslant 0} = 1\,. \tag{6}$$

In the absence of a PGW, the plasma is at rest, i.e.,

$$\psi(u,v)_{|u\leqslant 0} = \psi_0(x^1)\,; \tag{7}$$

$$p_{|u\leqslant 0} = p_0(x^1)\,; \qquad \varepsilon_{|u\leqslant 0} = \varepsilon_0(x^1)\,, \tag{8}$$

where $\psi_0\,, p_0\,, \varepsilon_0$ - are some given functions of the variable x^1;

$$v_{v|u\leqslant 0} = v_{v|u\leqslant 0} = \frac{1}{\sqrt{2}} \Longrightarrow -\left(\frac{\partial_u\psi}{\partial_v\psi}\right)_{|u\leqslant 0} = 1\,. \tag{9}$$

Thus, due to (9):

$$-\,(\partial_u\psi)_{|u\leqslant 0} = (\partial_v\psi)_{|u\leqslant 0}\,. \tag{10}$$

In the absence of a PGW, the energy density, pressure and magnetic field strength are identical everywhere:

$$\varepsilon_{|u\leqslant 0} = \varepsilon_0 = \mathrm{Const}\,; \qquad p_{|u\leqslant 0} = p_0 = \mathrm{Const}\,; \tag{11}$$

$$H_{|u\leqslant 0} = H_0 = \mathrm{Const} \Rightarrow \psi_{|u\leqslant 0} = \frac{1}{\sqrt{2}}H_0(v-u), \tag{12}$$

at $u \leqslant 0\,; \forall\ v\ \in \mathcal{R}$.

In Ref. [4] it has been shown that the plasma macroscopic parameters are the following:

$$v_u = \sqrt{-\frac{\partial_u Z}{\partial_v Z}}\,; \qquad v_v = \sqrt{-\frac{\partial_v Z}{\partial_u Z}}\,. \tag{13}$$

$$H^2 = -L^{-4} e^{2\beta} \partial_u Z \partial_v Z \,, \tag{14}$$

$$\varepsilon = L^{-2} \varepsilon_0 \sqrt{-\partial_u Z \partial_v Z} \tag{15}$$

where

$$Z = \frac{\sqrt{2}\psi}{H_0} \tag{16}$$

is a dimensionless function.

The function $Z(u, v)$ should satisfy a boundary condition on the spacelike hypersurface $\Sigma_r : u = v\,(x = 0)$.) Let us put, following Ref. [1], on this hypersurface:

$$H_{|u=v} = \text{Const.} \tag{17}$$

Then, taking into account the relation (see Ref. [1]):

$$H_2 = F_{13} = \frac{1}{\sqrt{2}}(F_{v3} - F_{u3}) = \frac{1}{\sqrt{2}}(\partial_v \psi - \partial_u \psi) \,. \tag{18}$$

we bring the boundary condition (17) to the form:

$$-\left(\partial_u Z \partial_v Z\right)_{|u=v} = 1 \,. \tag{19}$$

Taking into account these conditions, the set of RMHD equations is reduced to a single differential equation:

$$\begin{aligned} \partial_{uv} Z + \beta' \partial_v Z = {} & \frac{2\pi e^{-2\beta}}{H_0^2} \frac{dp}{d\varepsilon} (\varepsilon + p) \frac{(L^2)'}{\partial_u Z} - \\ & - \frac{\pi L^2}{H_0^2} e^{-2\beta} (\varepsilon + p) \Bigg\{ \left[\frac{\partial_{vv} Z}{(\partial_v Z)^2} + \frac{\partial_{uu} Z}{(\partial_u Z)^2} - 2 \frac{\partial_{uv} Z}{\partial_u Z \partial_v Z} \right] + \\ & + \frac{dp}{d\varepsilon} \left[\frac{\partial_{vv} Z}{(\partial_v Z)^2} + \frac{\partial_{uu} Z}{(\partial_u Z)^2} + 2 \frac{\partial_{uv} Z}{\partial_u Z \partial_v Z} \right] \Bigg\} \end{aligned} \tag{20}$$

which for a non-relativistic plasma becomes

$$\begin{aligned} \partial_{uv} Z + \beta' \partial_v Z = {} & -\frac{1}{4\alpha^2} e^{-2\beta} \sqrt{-\partial_v Z \partial_u Z} \times \\ & \times \left[\frac{\partial_{vv} Z}{(\partial_v Z)^2} + \frac{\partial_{uu} Z}{(\partial_u Z)^2} - 2 \frac{\partial_{uv} Z}{\partial_u Z \partial_v Z} \right] . \end{aligned} \tag{21}$$

(for details see Ref. [5]).

3 Search for a solution in a weak gravitational wave

Since the GMSW is formed in a highly magnetized plasma, in what follows we shall suppose that α (2) is a large parameter (see the condition (3)). Therefore,

Eq. (21), being expanded in the small dimensionless parameter of the problem α^{-1}, takes the form:

$$\partial_{uv} Z + \beta' \partial_v Z = 0 \tag{22}$$

Solving it, we obtain the first approximation, satisfying the conditions indicated above:

$$Z_0 = (v-u)\, e^{-\beta} - 2\int_0^u \sinh(\beta(u))\, du \tag{23}$$

The magnetic field behaviour corresponding to the solution (23) is shown in Fig.1:.

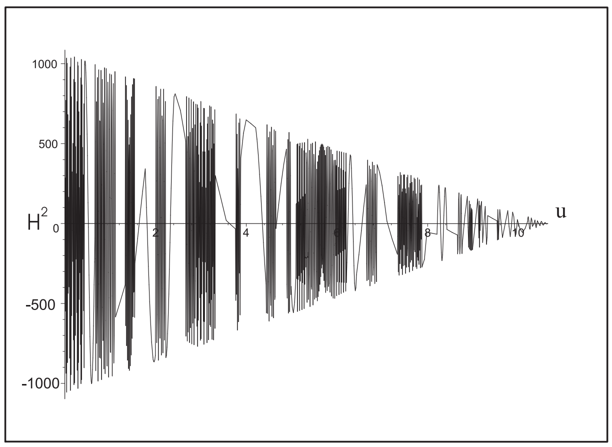


Figure 1: *Dependence of the magnetic field intensity on the retarded time calculated according to the solution (23).*

To verify the linear approximation applicability, we substitute the solution found to the r.h.s. of Eq. (21) and obtain:

$$\partial_{uv} Z + \beta' \partial_v Z = \Phi(u,v), \tag{24}$$

where $\Phi(u,v)$ is a very unwieldy expression. It is, however, easy to see that the radicand $-\partial_u Z_0 \partial_v Z_0$ can be negative on some surface $\Sigma(u,v)$ (an analysis shows that it is $\partial_u Z_0$ that changes its sign). It means that near this surface the linear solution of Eq. (23) becomes inapplicable. Supposing that the GW amplitude is everywhere small,

$$|\beta(u)| \ll 1 \tag{25}$$

and decomposing the radicand at the r.h.s. of Eq. (24) in series with respect to β, we obtain the equation of this surface:

$$\Sigma(u,v): \quad (v-u)\beta' + \beta + 1 = 0, \tag{26}$$

which shows that this surface exists at sufficiently large values of the variable $(v-u)=\sqrt{2}x$, i.e., it is far from the boundary:

$$|v-u| \sim \frac{1}{\beta_0\omega}, \tag{27}$$

where β_0 is the GW amplitude and ω its frequency.

Near the surfaces $\Sigma(u,v)$:

$$1+\beta'(v-u)=\sigma \ll 1, \tag{28}$$

whence, putting $\beta(u)=\beta_0(1-\cos\omega u)$, we obtain the equation of the surface $\Sigma(u,v)$, resolved explicitly with respect to retarded time:

$$v=\frac{-1+\sigma}{\beta_0\omega\sin(\omega u)+u}; \tag{29}$$

The ranges in which Eq. (29) holds are represented by narrow parabolas which have almost flat vertices lying near the straight line $u=v+a\,(a=\mathrm{Const})$. Their qualitative behaviour is depicted in Fig. 2:

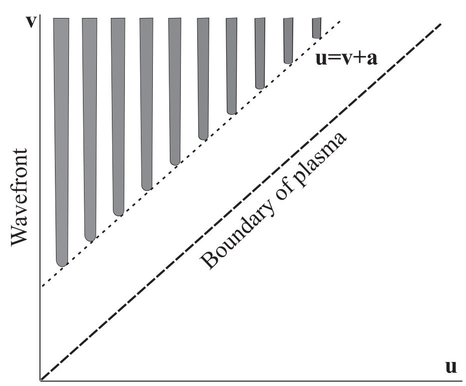


Figure 2: *Ranges of violation of the linear approximation of Eq. (22)*

A detailed image of the vertex of one of the parabolas is shown in Fig. 3.

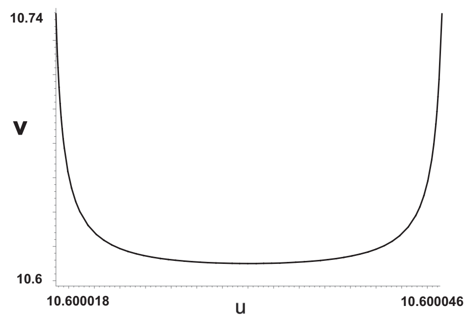


Figure 3: *The dependence v(u) inside one of ranges, shown in Fig. 2.*

Far from this surface, Eq. (24) is integrated, and we obtain the first-order correction:

$$Z_1 = (v - u)\, e^{-\beta} \zeta + 2 \int_0^u \zeta \cosh(\beta(u))\, du \tag{30}$$

where

$$\zeta = \int_0^u \Phi(u) e^{\beta(u)}\, du. \tag{31}$$

Since an analytical solution to Eq. (21) cannot be found, while a direct application of numerical methods faces considerable difficulties, related to large values of the derivatives near singular points, we have used, for a numerical solution of Eq. (21), the symmetric reflection method, approximating the solution behaviour near a singular point with a symmetric parabola.

Comparing the resulting numerical solution of Eq. (21) near singular points with the analytical solution (23) of Eq. (22), we conclude that, despite the difficulties indicated above, these solutions almost coincide. An illustration is given in Fig. 4:

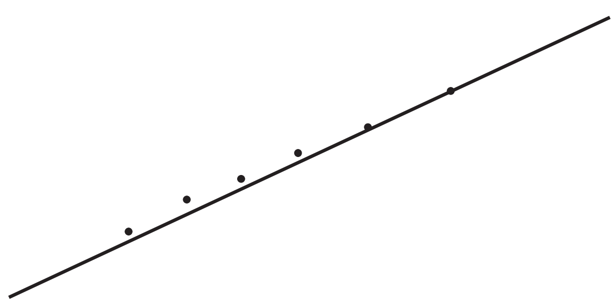
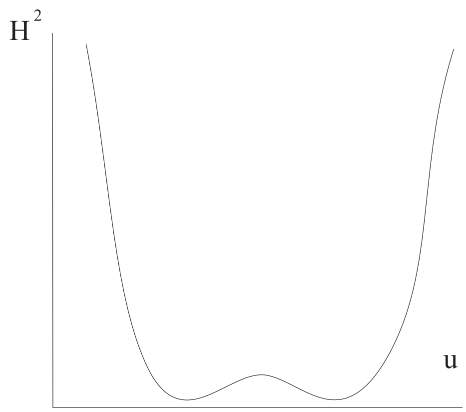


Figure 4: *Comparison of the numerical and analytical solutions at small scale inside a range shown in 2: the straight line represents an analytical solution, the points give a numerical solution.*

5 shows a qualitative pattern of the magnetic field strength squared near a singular point. As a result, taking into account the nonlinearity of Eq. (21) near singular2 points reduces, basically, to a cut-off of the lower range ($H^2 < 0$) in the graph 1 and to formation of a plateau of the function H^2 near zero. Let us remark that a similar behaviour of magnetoactive plasma inside a nonlinear range was also established in Ref. [4].



. 5. *Magneticfield behaviour near singular points*

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Let us also remark that, in a weak gravitational wave ($\beta \ll 1$), the condition $\partial_v Z_0 \approx 1$ is satisfied, and therefore Eq. (21) can be reduced to the form of a first-order quasi-linear partial differential equation:

$$\partial_u \varphi * \varphi^{-3/2} + 4\alpha^2 \partial_v \varphi = \beta' \tag{32}$$

where $\varphi = -\partial_u Z$.

It is easy to obtain the formal common solution of this equation, however, since it is impossible to find this solution explicitly, it is also impossible to find the function $Z(u, v)$ satisfying the required initial and boundary conditions.

References

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